Lecture 10
Single bunch longitudinal instabilities

January 24, 2019

Lecture outline

- Longitudinal motion of particles in bunch.
- The Vlasov equation for the distribution function.
- Keil-Schnell stability criterium for coasting beam.
- Haissinski equation for the beam equilibrium.

Long-range and short-range wake fields

In L8 and L9 we studied the long-range wake fields. We modeled the beam as a set of bunches, with each bunch represented by a single large "point-like" charge. The long-range wake fields couple the motion of different bunches.

In this lecture, we will study effects of a short-range *longitudinal* wake (we studied the short-range transverse wake in L7 for BBU). A short-range wake field is one that extends only over the length of a single bunch. (In the frequency domain, this corresponds to high frequencies, $\omega \gtrsim c/\sigma_z$).

To understand the effects of short-range wake fields, we have to consider the "internal" dynamics of individual bunches. We will model the bunch as a charge distribution, and try to work out how the distribution function evolves over time, in the presence of a wake field.

We will use the fact that longitudinal instabilities are slow and evolve on a time scale much larger than the revolution period T.

Longitudinal dynamics with wakes

For the longitudinal wake *per unit path*, w_ℓ, $-cq^2 w_\ell$ is the energy change per unit time (see Eq. (3.1)), so

$$\dot{\eta} = \frac{1}{\alpha}\frac{\omega_{s0}^2}{c}z - \frac{q^2}{\gamma mc}W_\ell \tag{10.1}$$

where W_ℓ is the wake in the bunch. We find (see Eq. (9.4))

$$\ddot{z} + \omega_{s0}^2 z = \frac{q^2\alpha}{\gamma m}W_\ell \tag{10.2}$$

The equation for η can be written as

$$\dot{\eta} = K(z,t) \tag{10.3}$$

where ($q \rightarrow e$)

$$K(z,t) = \frac{\omega_{s0}^2}{\alpha c}z - \frac{e^2}{\gamma mc}\int_z^\infty dz' n(z',t) w_\ell(z'-z) \tag{10.4}$$

(if the wake does not vanish for negative z then the integration goes from minus infinity). In Eq. (10.4) $n(z,t) = N\lambda(z,t)$ is the linear beam density (number of particles per unit length), $\int_{-\infty}^{\infty} n(s,t)ds = N$, where N is the number of particles in the bunch. The second term in K gives the energy change resulting from the wake fields.

Longitudinal Hamiltonian

The wake for the whole ring is equal to Cw_ℓ where C is the ring circumference.
Eqs. (10.3) and (10.4) can be also written with the help of a Hamiltonian,

$$\begin{aligned} H(z, -\eta, t) &= \frac{1}{2} c\alpha\eta^2 + V(z,t) \\ &= \frac{1}{2} c\alpha\eta^2 + \frac{\omega_{s0}^2}{2\alpha c} z^2 - \frac{e^2}{\gamma mc} \int_0^z dz' \int_{z'}^{\infty} dz'' n(z'', t) w_\ell(z'' - z') \end{aligned} \qquad (10.5)$$

in which z plays a role of a coordinate, and $-\eta$ is the conjugate momentum.
We have

$$\dot{z} = \frac{\partial H}{\partial(-\eta)}, \qquad \dot{\eta} = \frac{\partial H}{\partial z} \qquad (10.6)$$

Longitudinal Hamiltonian

Exchanging the order of integration over z' and z'' in Eq. (10.5) we can write the Hamiltonian as

$$H(z, -\eta, t) = \frac{1}{2}c\alpha\eta^2 + \frac{\omega_{z0}^2}{2\alpha c}z^2 + \frac{e^2}{\gamma mc}\int_z^\infty dz' n(z', t) S(z' - z) \qquad (10.7)$$

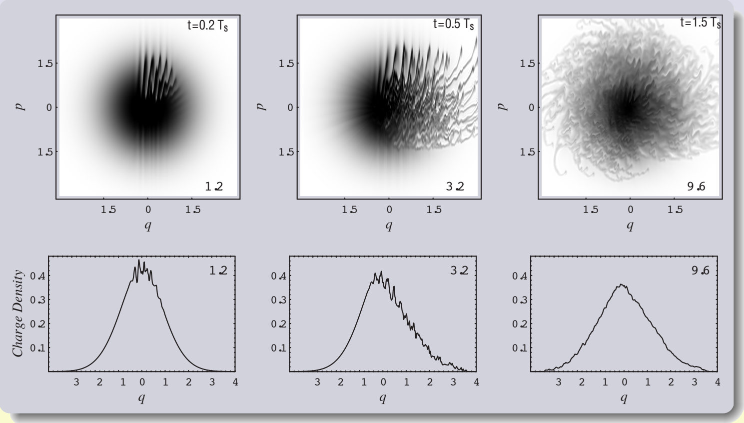
where

$$S(z) = \int_0^z ds\, w_\ell(s)$$

We will need this formulation when we discuss the beam equilibrium later.

Distribution function f

We will use a distribution function $f(z, \eta, t)$ of the particles. The quantity $dN = f(z, \eta, t)dz\,d\eta$ is the number of particles in the infinitesimally small area of the phase space $dz\,d\eta$. Integrating over η gives the particle density $n(z,t) = \int_{-\infty}^{\infty} f(z, \eta, t)d\eta$. An example of numerical simulations of f.



From Venturini et al., PRAB, **8**, 014202 (2005).

The Vlasov equation

This distribution function satisfies the Vlasov equation (9.21)

$$\frac{\partial f}{\partial t} + \dot{z}\frac{\partial f}{\partial z} + \dot{\eta}\frac{\partial f}{\partial \eta} = 0 \tag{10.8}$$

Substituting $\dot{z}$ and $\dot{\eta}$ from (10.3) we obtain

$$\frac{\partial f}{\partial t} - c\alpha\eta\frac{\partial f}{\partial z} + K(z,t)\frac{\partial f}{\partial \eta} = 0 \tag{10.9}$$

If we want to take into account damping and diffusion due to synchrotron radiation, then

$$\frac{\partial f}{\partial t} - c\alpha\eta\frac{\partial f}{\partial z} + K(z,t)\frac{\partial f}{\partial \eta} = R[f]$$

where R is a (differential or integral) operator acting on f (see[29]). In what follows we set $R = 0$.

[29] G. Stupakov and G. Penn. *Classical Mechanics and Electromagnetism in Accelerator Physics, Springer, 2018.*

Coasting beam approximation

The simplest stability problem that can be solved analytically with the Vlasov equation is stability of a coasting beam. For a coasting beam we can use the Vlasov equation (10.9) putting $\omega_{s0} = 0$ in (10.4).

Before addressing stability, one has to define an equilibrium solution, that is the one which does not depend on time. We start from the equilibrium solution in which f does not depend on z (coasting) and t, $f = f_0(\eta)$. Correspondingly, the beam (linear) density is constant $n = n_0$. To satisfy (10.9) we need

$$K = -\frac{e^2 n_0}{\gamma mc} \int_z^\infty dz' \, w_\ell(z' - z) = 0$$

or

$$\int_0^\infty w_\ell(s) ds = 0$$

As we discussed earlier, this is an important property of the longitudinal wakes.

Longitudinal instability for a coasting beam

We now assume $f = f_0(\eta) + f_1(z, \eta, t)$ and $n = n_0 + n_1$ with $|f_1| \ll f_0$ and $|n_1| \ll n_0$ (remember that $n(z,t) = \int\limits_{-\infty}^{\infty} f(z,\eta,t) d\eta$). We *linearize* Eq. (10.9) keeping only first order terms in f_1 and n_1. We then assume $f_1, n_1 \propto \exp(-i\omega t + ikz)$, where ω is the frequency and $k = 2\pi/\lambda$ is the wavenumber of the perturbation,

$$- i\omega f_1 - ikc\alpha\eta f_1 - \frac{e^2}{\gamma mc}\frac{df_0}{d\eta} n_1 \int_z^\infty dz' e^{-ikz+ikz'} w_\ell(z'-z) = 0$$

The last integral is equal to $cZ_\ell(ck)$ where Z_ℓ is the longitudinal impedance per unit length. This gives

$$- i(\omega + kc\alpha\eta) f_1 = \frac{e^2}{\gamma m}\frac{df_0}{d\eta} n_1 Z_\ell(ck)$$

Longitudinal instability for a coasting beam

from which we find f_1

$$f_1 = i\frac{e^2}{\gamma m}\frac{df_0}{d\eta}n_1 Z_\ell(ck)\frac{1}{\omega + kc\alpha\eta}$$

We now use $n_1 = \int f_1 d\eta$. Integrating over η gives the *dispersion relation* we obtain

$$i\frac{e^2}{\gamma m}Z_\ell(ck)\int_{-\infty}^{\infty} d\eta \frac{df_0/d\eta}{\omega + kc\alpha\eta} = 1 \tag{10.10}$$

Consider first the case of a cold beam, $f_0 = n_0\delta(\eta)$. Integrating by parts gives

$$n_0\int_{-\infty}^{\infty} d\eta \frac{d\delta(\eta)/d\eta}{\omega + kc\alpha\eta} = c\alpha k n_0\int_{-\infty}^{\infty} d\eta \frac{\delta(\eta)}{(\omega + kc\alpha\eta)^2} = \frac{c\alpha k n_0}{\omega^2}$$

Cold coasting beam is (almost) always unstable

This gives the dispersion relation

$$\omega^2 = i\frac{\alpha e^2 c n_0 k}{\gamma m} Z_\ell(ck) = i\frac{\alpha e I k}{\gamma m} Z_\ell(ck) \tag{10.11}$$

where we have used the beam current $I = ecn_0$. Since Z_ℓ is a complex number, this means that the beam is unstable (unless Z_ℓ is purely imaginary with $\mathrm{Im}\, Z_\ell < 0$, [and the ring is above the transition, $\alpha > 0$], which is the case of an inductive impedance (6.2), $Z_\ell = -i\omega L$).

Example of instability: the resistive wake (6.1) with $Z_\ell = R$:

$$\omega_0(k) = \pm(1+i)\sqrt{\frac{\alpha e I k R}{2\gamma m}} \tag{10.12}$$

Shorter wavelengths (larger values of k) have a larger growth rate.

Finite energy spread and Landau damping

With finite energy spread, we need to solve

$$\frac{e^2}{m\gamma} Z_\ell(ck) = \left(i \int_{-\infty}^{\infty} d\eta \frac{df_0/d\eta}{\omega + kc\alpha\eta} \right)^{-1} \tag{10.13}$$

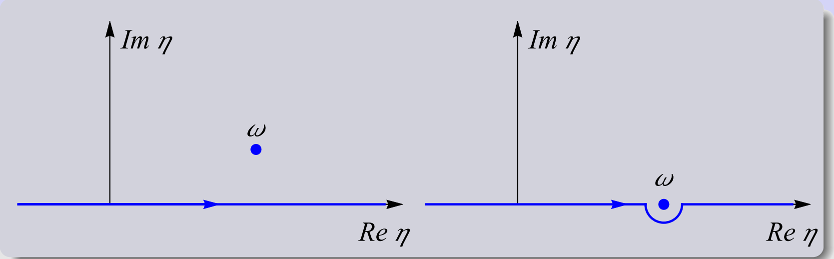
and find the *dispersion relation* $\omega(k)$. If $\mathrm{Im}\,\omega(k) > 0$ we have an instability. For a given value of k we need to be able to calculate

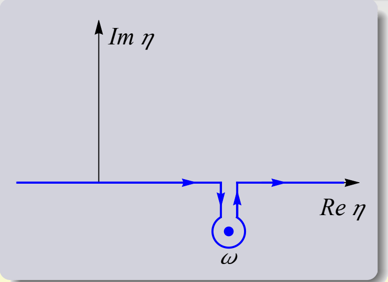
$$\int_{-\infty}^{\infty} d\eta \frac{df_0/d\eta}{\omega + kc\alpha\eta} \tag{10.14}$$

for every value of ω, including real ω. However, for real ω there is a singularity in the integrand. How do we treat it?

This problem has been solved by L. Landau in 1946, for plasma oscillations. He showed that the correct solution can be obtained using the Laplace transform of the Vlasov equation (instead of the Fourier transform in ω) with initial conditions. His result: one can use (10.14) in the upper half plane of complex ω. For the lower half-plane, or real *omega*, one should change the integration path in the complex plane of η.

Integration path in complex plane η





Note that we have to redefine our distribution function f_0 as a function of the complex variable η (that is, to *analytically extend* it to the complex plane η). For a Gaussian distribution function $\propto e^{-\eta^2/2\sigma_\eta^2}$ this is not a problem.

Finite energy spread and Landau damping

Some authors like to draw a stability diagram where the axes are the real and imaginary parts of the right hand side of Eq. (10.13)[30]. Then they infer the stability condition from this diagram. With modern computers this does not make much sense—one just need to solve this equation numerically scanning the k values. This can be done for a given distribution function $f_0(\eta)$ and impedance Z_ℓ.

[30] See lecture notes by A. Wolski.

Landau damping for resistive impedance

Two lines of Mathematica code solve the problem of the stability with resistive impedance R.

```
g[a_] = Integrate[(e^(-x^2/2) x)/(x + a), {x, -Infinity, Infinity}, Assumptions → Im[a] ≠ 0] // PowerExpand;

roots = Table[{y, x /. FindRoot[ⅈ/Sqrt[2 π] g[x] == -y, {x, 1}]}, {y, .01, 2, .01}];

pl1 = ListPlot[{#[[1]], Re[#[[2]]]/Sqrt[1/(2 #[[1]])]} & /@ roots, PlotStyle → Red];
pl2 = ListPlot[{#[[1]], Im[#[[2]]]/Sqrt[1/(2 #[[1]])]} & /@ roots, PlotStyle → Blue];
Show[pl1, pl2, Plot[0, {y, .0, 1}, PlotStyle -> Dashed], DisplayFunction → $DisplayFunction,
 FrameLabel -> {"k/k0", "Reω/Reω0,Imω/Imω0"},
 PlotRange → {{0, 1}, All}]
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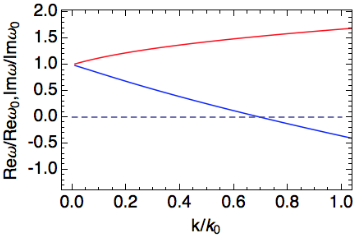
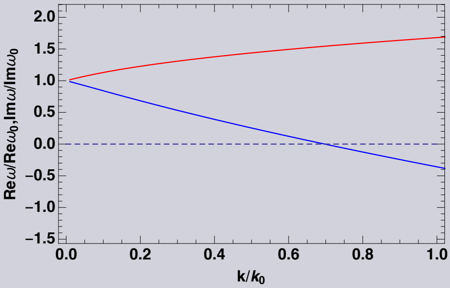


Landau damping for resistive impedance

Plot of $\mathrm{Re}\,\omega$ (red) and $\mathrm{Im}\,\omega$ (blue) normalized by the cold limit ω_0 (see (10.12)) as a function of k normalized by k_0



$$k_0 = \frac{e^2 R n_0}{\gamma m \sigma_\eta^2 c \alpha} \qquad (10.15)$$

The instability is suppressed for $k > 0.7 k_0$.

The stabilizing mechanism that suppresses the instability at large values of k is called the *Landau damping*. You can read a detailed explanation of Landau damping mechanism in A. Chao's book, Section 5.1.

General criterion for microwave instability

Analysis of several models of impedance show that the stability criterion is approximately valid if we replace R by $|Z(ck)|$

$$k \gtrsim \frac{e^2 |Z(ck)| n_0}{\gamma m \sigma_\eta^2 c \alpha} \tag{10.16}$$

or

$$\sigma_\eta^2 \gtrsim \frac{e^2 n_0}{\gamma m k c \alpha} |Z_\ell(ck)| \tag{10.17}$$

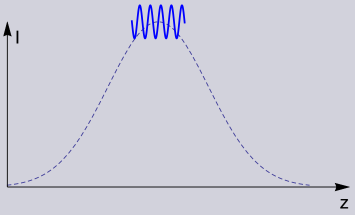
It is often written in a different way. The impedance in Eq. (10.17) is per unit length. If we use the impedance of the ring, Z°, then $Z_\ell = Z^\circ / cT$.

$$\sigma_\eta^2 \gtrsim \frac{e^2 n_0}{m k c^2 T \gamma \alpha} |Z^\circ(ck)| \tag{10.18}$$

The quantity $1/cTk$ is denoted as $1/2\pi n$, where $n = ck/\omega_{\mathrm{rev}} = Ck/2\pi$. Noting that $c e n_0 = I$, we obtain

$$2\pi\alpha(\gamma m c^2)\sigma_\eta^2 \gtrsim eI \frac{|Z^\circ(n)|}{n} \tag{10.19}$$

Keil-Schnell-Boussard criterion



We derived this criterion for a coasting beam, but it can also be applied to a bunched beam if we consider perturbations with the wavelength $\lambda/2\pi \ll \sigma_z$. Then locally the beam looks like a coasting one. For the beam current I we should use the peak current in the bunch (not the averaged current in the ring).

If this criterion is used for a bunched beam, it gives a crude estimate for the stability requirements of the beam. In this context it is often called the Keil-Schnell or Keil-Schnell-Boussard criterion. The instability is often referred to as the *microwave instability*. It is associated with *turbulent bunch lengthening*.

Bunch equilibrium without wake

A strong longitudinal wake in a bunch not only can cause beam instability, but also modifies its equilibrium. Here we will derive an equation that describes the beam equilibrium with account of the wake.
Let us first consider the equilibrium without a wake. Start from the Hamiltonian (10.5) with $w_\ell = 0$,

$$H(z, -\eta) = \frac{1}{2} c\alpha\eta^2 + V(z) = \frac{1}{2} c\alpha\eta^2 + \frac{\omega_{s0}^2}{2\alpha c} z^2 \tag{10.20}$$

We know that the distribution function $f(\eta, z)$ is constant along the trajectories. Since the Hamiltonian does not depend on time, it is constant along the trajectories. It is reasonable to assume that f is a function of H only. On the other hand, it is known that in electron rings, the longitudinal distribution function is Gaussian

$$f(\eta, z) = A \exp\left(-\frac{\eta^2}{2\sigma_\eta^2} - \frac{z^2}{2\sigma_{z0}^2}\right) \tag{10.21}$$

where A is the normalization constant.

Bunch equilibrium without wake

This is possible if

$$f(\eta, z) = A \exp\left(-\frac{H}{c\alpha\sigma_\eta^2}\right) = A \exp\left(-\frac{\eta^2}{2\sigma_\eta^2} - \frac{\omega_{s0}^2 z^2}{2\alpha^2 c^2 \sigma_\eta^2}\right) \qquad (10.22)$$

We also found that $\sigma_{z0} = \alpha\sigma_\eta c/\omega_{s0}$.

Equilibrium with wake

It turns out that with account of the wake the equilibrium is given by the same equation, which is now called the *Haïssinski* equation

$$f = \mathcal{Z} \exp\left(-\frac{H}{c\alpha\sigma_\eta^2}\right) = \mathcal{Z} \exp\left(-\frac{\eta^2}{2\sigma_\eta^2} - \frac{V(z)}{c\alpha\sigma_\eta^2}\right) \tag{10.23}$$

But now this should be solved self-consistently because V is a functional of $n(z)$

$$\begin{aligned} V(z) &= \frac{\omega_{s0}^2}{2\alpha c} z^2 + \frac{e^2}{mc\gamma} \int_z^\infty dz' n(z', t) S(z' - z) \\ &= \frac{\omega_{s0}^2}{2\alpha c} z^2 + \frac{e^2}{mc\gamma} \int_z^\infty dz' S(z' - z) \int d\eta f(z', \eta) \end{aligned} \tag{10.24}$$

Numerical solution of the *Haïssinski* equation

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Editors' Suggestion

Numerical solution of the Haïssinski equation for the equilibrium state of a stored electron beam

Robert Warnock*

SLAC National Accelerator Laboratory, Stanford University, Menlo Park, California 94025, USA and Department of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico 87131, USA

Karl Bane†

SLAC National Accelerator Laboratory, Stanford University, Menlo Park, California 94025, USA

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The longitudinal charge density of an electron beam in its equilibrium state is given by the solution of the Haïssinski equation, which provides a stationary solution of the Vlasov-Fokker-Planck equation. The physical input is the longitudinal wake potential. We formulate the Haïssinski equation as a nonlinear integral equation with the normalization integral stated as a functional of the solution. This equation can be solved in a simple way by the matrix version of Newtons's iteration, beginning with the Gaussian as a first guess. We illustrate for several quasirealistic wake potentials. Convergence is extremely robust, even at currents much higher than nominal for the storage rings considered. The method overcomes limitations of earlier procedures, and provides the convenience of automatic normalization of the solution.

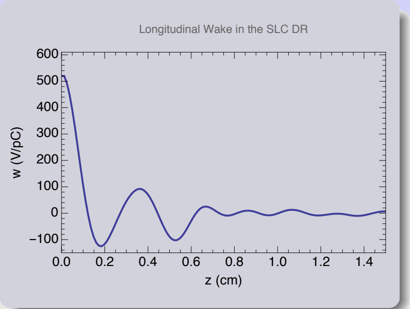
DOI: 10.1103/PhysRevAccelBeams.21.124401

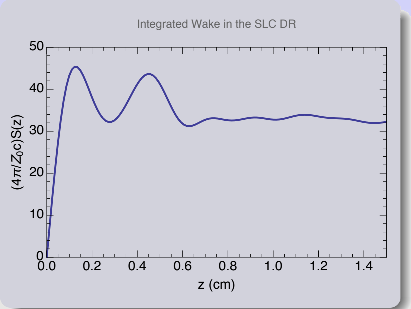
Solution of Haïssinski equation—SLCDR

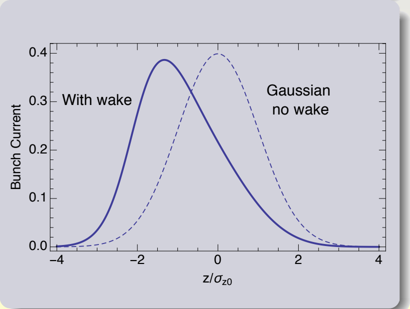
We consider here an example of the SLC damping ring, whose stability was studied in great detail by K. Bane from SLAC.

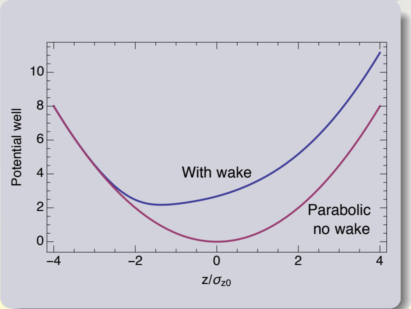
Parameter	Value	Units
Energy	1.15	GeV
N_p	2×10^{10}	
T_0	118	ns
$\omega_0/2\pi$	99	kHz
σ_η	7×10^{-4}	
σ_{z0}	0.5	cm

Wake and and the beam equilibrium

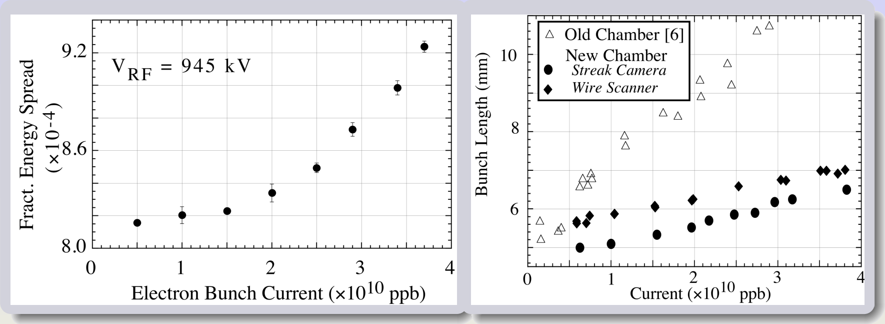








Observations of the microwave instability



From K. Bane et al. SLAC-PUB-95-6894 (1995).

Synchrotron frequency depends on the amplitude

One of the important consequences of Haïssinski equilibrium is that the synchrotron frequency is not constant any more — it depends on the amplitude of the oscillations in the potential well,

$$\omega_s(J) = \omega_{s0} + \Delta\omega_s(J), \tag{10.25}$$

where J is the "action" variable for the longitudinal motion. This occurs because the potential well for the synchrotron oscillations is not parabolic any more.